

Bäcklund transformations for the Camassa-Holm equation

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Abstract

The Bäcklund transformation (BT) for the Camassa-Holm (CH) equation is presented and discussed. Unlike the vast majority of BTs studied in the past, for CH the transformation acts on both the dependent and (one of) the independent variables. Superposition principles are given for the action of double BTs on the variables of the CH and the potential CH equations. Applications of the BT and its superposition principles are presented, specifically the construction of travelling wave solutions, a new method to construct multi-soliton, multi-cuspon and soliton-cuspon solutions, and a derivation of generating functions for the local symmetries and conservation laws of the CH hierarchy.

1 Introduction

The original Bäcklund transformation (BT) arose in the context of differential geometry of surfaces in the 1880s [3]. In the modern era, BTs have been recognized as playing a central role in the theory of integrable differential equations [34, 59, 58]. Their primary application is as a method to generate explicit solutions, exploiting the so-called superposition principle, an algebraic rule to “combine” two solutions obtained by BTs (from a given initial solution). However, in recent work [55] we have also shown how to derive local symmetries and conservation laws directly from BTs. There is also a deep relationship between BTs and the associated linear systems of integrable equations.

The Camassa-Holm (CH) equation [9, 10] is by now recognized as one of the archetypes of integrable equations. It has (weak) “peakon” solutions — solitary waves with discontinuous first derivative at their crest — and numerous other types of travelling wave solution, including solitons (smooth solitary waves), cuspons and various periodic structures [40, 41, 7, 8, 33, 47, 48, 50, 37, 54]. The integrability of the CH equation was already firmly

established in [9], where a Lax pair and a bihamiltonian structure were given, and much further evidence for this has accumulated since then. There is an inverse scattering formalism [13, 14], explicit formulas can be found for multipeakon, multisoliton, multicusp and soliton-cuspon solutions [60, 4, 5, 6, 22, 31, 39, 49, 18, 38, 43, 53, 44, 51, 52, 19, 63] there are an infinite number of local conservation laws [23, 56, 57, 28, 25, 27, 35, 11, 30, 24], and there is a rich algebra of symmetries [56, 57, 28, 25, 27, 24]. Other significant works on CH include studies of the stability of peakon and other exact solutions [15, 16, 17, 36] and interesting numerical studies [32, 45, 21, 12].

The aim of this paper is to fully explore the theory of the BT for the CH equation. In [60], one of us constructed a BT for the associated CH (aCH) equation, an equation related by a (field dependent) change of coordinates to the CH equation, and used this to construct some solutions of CH which could be regarded as superpositions of 2 travelling waves. However, this work was incomplete; an integration was required to reconstruct a solution of CH from a solution of aCH, which, in general, could not be done explicitly, severely limiting applicability. In the current paper we resolve this and other problems. The BT of CH differs from standard ones (for example, those of KdV and Sine-Gordon) in that *it involves a transformation of both the dependent and one of the independent variables*. However, remarkably, there is a nonlinear superposition principle for both of these transformations, which we develop and apply to the generation of multisoliton, multicusp and soliton-cuspon solutions, as well as to the derivation of symmetries and conservation laws for CH. The action of the BT on both dependent and independent variables is not unique to CH; a similar situation exists for the Dym equation, which also exhibits nonanalytic solitons [62, 61].

The structure of this paper is as follows: In section 2 we recap the known results for the aCH equation. In section 3 we use them to derive the BT for CH. Section 4 discusses the various forms of superposition principle. In section 5 we use the BT to obtain travelling wave solutions. The BT is used to construct soliton and cuspon solutions from which the standard peakon solutions can be obtained in a certain limit. Alas it does not seem to give a direct construction of peakons. However, various other unphysical solutions are also obtained. In section 6 we use the superposition principle to obtain cuspon-cuspon, soliton-soliton and cuspon-soliton solutions. In section 7, following [55], we use the BT to construct the conservation laws and symmetries of CH. Section 8 contains some concluding remarks.

2 Previous results

The Camassa-Holm equation (CH) [9] is

$$m_t + 2u_x m + u m_x = 0, \quad m = u - u_{xx}, \quad (1)$$

or equivalently

$$u_t - u_{txx} + 3uu_x - uu_{xxx} - 2u_x u_{xx} = 0. \quad (2)$$

By translating u and performing a Galilean transformation $x \rightarrow x - ct$ it is possible to introduce linear transport and linear dispersion terms into the equation, see for example [20]. All the results we present here can be generalized for the full class of equations considered in [20].

Writing $u = v_x$ and integrating once, we obtain the potential Camassa-Holm equation (pCH)

$$v_t - v_{txx} + \frac{3}{2}v_x^2 - v_x v_{xxx} - \frac{1}{2}v_{xx}^2 = 0 ,$$

or, equivalently,

$$n_t + \frac{1}{2}v_x^2 + v_x n_x - \frac{1}{2}(v - n)^2 = 0 , \quad n = v - v_{xx} .$$

Evidently n is a potential for m , $m = n_x$.

In [60] equation (1), under the assumption $m > 0$, was transformed to the associated Camassa-Holm equation (aCH)

$$2p_\tau = -p^2 u_\xi , \quad u = -\frac{p}{2} \left(\frac{p_\tau}{p} \right)_\xi + p^2 ,$$

with the help of transformation

$$p = \sqrt{m} , \quad d\xi = \frac{1}{2}pdx - \frac{1}{2}pudt , \quad \tau = t. \quad (3)$$

This transformation implies

$$\frac{\partial x}{\partial \xi} = \frac{2}{p} , \quad \frac{\partial x}{\partial \tau} = u , \quad \frac{\partial t}{\partial \xi} = 0 , \quad \frac{\partial t}{\partial \tau} = 1 . \quad (4)$$

A BT for aCH was found in [60]:

$$p \rightarrow p - s_\xi , \quad u \rightarrow u + \frac{2s_\tau}{p(p - s_\xi)} , \quad (5)$$

where s satisfies

$$s_\xi = -\frac{s^2}{p\alpha} + \frac{\alpha}{p} + p , \quad (6)$$

$$s_\tau = -s^2 + \frac{p_\tau}{p}s + \alpha(\alpha + u) . \quad (7)$$

The following nonlinear superposition principle was also given:

$$p \rightarrow p - \left(\frac{(\alpha - \beta)(\alpha\beta - s_\alpha s_\beta)}{\beta s_\alpha - \alpha s_\beta} \right)_\xi , \quad (8)$$

where s_α, s_β are the solutions of (6,7) with parameters α and β respectively.

In [55] the BT was used to find an infinite number of symmetries for aCH. These are given by the generating symmetry $X = Q^p \frac{\partial}{\partial p} + Q^u \frac{\partial}{\partial u}$ where

$$Q^p = \frac{p(s_\alpha^{(1)} + s_\alpha^{(2)})}{\alpha(s_\alpha^{(1)} - s_\alpha^{(2)})} , \quad Q^u = -\frac{2s_\alpha^{(1)} + 2s_\alpha^{(2)} + pu_\xi}{s_\alpha^{(1)} - s_\alpha^{(2)}} . \quad (9)$$

Here $s_\alpha^{(1)}, s_\alpha^{(2)}$ are two different solutions of (6,7) for the same parameter α . This symmetry depends upon α ; expansion in a (formal) power series in α gives the infinite hierarchy of symmetries.

3 The Bäcklund transformation for the Camassa-Holm equation

In this section we obtain the BT for CH and pCH from the BT for aCH. With the help of (4) we write the BT (5),(6),(7) as

$$u \rightarrow u - 2\alpha - \frac{2\alpha(u_x s - \alpha u)}{s^2 - \alpha^2} , \quad (10)$$

where s satisfies

$$s_x = -\frac{s^2}{2\alpha} + \frac{1}{2}(m + \alpha) , \quad (11)$$

$$s_t = -s^2 \left(1 - \frac{u}{2\alpha}\right) - u_x s + \frac{1}{2}(2\alpha^2 + \alpha u - um) . \quad (12)$$

This system for s is equivalent to the Lax pair for CH. Note (12) can be simplified with the help of (11) and (1) to

$$s_t = \alpha u_{xx} + 2\alpha s_x - u s_x - u_x s . \quad (13)$$

In light of (4) the BT for CH must also involve the independent variable x . Using the first equation in (4), the change of the independent variable is

$$\begin{aligned} x_{\text{new}} - x &= \int \left(\frac{2}{p_{\text{new}}} - \frac{2}{p} \right) d\xi \\ &= \int \left(\frac{2}{p - s_\xi} - \frac{2}{p} \right) d\xi \\ &= \int \frac{2s_\xi}{p(p - s_\xi)} d\xi \\ &= \int \frac{2ds}{\frac{s^2}{\alpha} - \alpha} \\ &= \ln \left| \frac{s - \alpha}{s + \alpha} \right| + f(\tau) . \end{aligned}$$

In moving from the third to the fourth line here the formula for s_ξ in (6) is used in the denominator but not in the numerator. The integration leaves undetermined an arbitrary function $f(\tau)$. Using the second equation in (4) it is straightforward to show this must be a constant, which can be taken, without loss of generality, to be zero. Thus the effect of the BT on the independent coordinates is

$$x \rightarrow x + \ln \left| \frac{s - \alpha}{s + \alpha} \right| , \quad t \rightarrow t. \quad (14)$$

There is no guarantee that this mapping will be a bijection. We will see later an example in which the BT generates several solutions out of one, in the case that this mapping is not 1 to 1.

Using (5) and (6) it is straightforward to write down the BT for the field p

$$p \rightarrow \frac{s^2 - \alpha^2}{\alpha p} \quad (15)$$

and hence also for the field $m = p^2 = u - u_{xx}$

$$m \rightarrow \frac{(s^2 - \alpha^2)^2}{\alpha^2 m}. \quad (16)$$

Further calculations give the action of the BT for the pCH fields v (satisfying $u = v_x$) and $n = v - v_{xx}$:

$$n \rightarrow n - 2s, \quad (17)$$

$$v \rightarrow v + \frac{2\alpha(\alpha u_x - us)}{s^2 - \alpha^2}. \quad (18)$$

As mentioned above, the BT can be generalized for the full family of equations from [20]

$$c_1 u_x + c_2 u_{xxx} + c_3(u_t + 3uu_x) = c_4(u_{txx} + uu_{xxx} + 2u_x u_{xx}), \quad (19)$$

where c_1, c_2, c_3, c_4 are constants. (This generalized equation is referred to in [26, 64] as the “CH-r equation”.) The BT is

$$x \rightarrow x + \sqrt{\frac{c_4}{c_3}} \ln \left| \frac{s\sqrt{c_4} - \alpha\sqrt{c_3}}{s\sqrt{c_4} + \alpha\sqrt{c_3}} \right|, \quad u \rightarrow u - 2\alpha - \frac{2c_3\alpha^2 u - 2c_4\alpha s u_x + c_1\alpha^2 + c_2 s^2}{c_3\alpha^2 - c_4 s^2}.$$

Here s satisfies

$$s_x = -\frac{s^2}{2\alpha} + \frac{\alpha}{2} \frac{2c_3 u - 2c_4 u_{xx} + 2c_3\alpha + c_1}{2c_4\alpha - c_2}, \quad (20)$$

$$s_t = \alpha u_{xx} + 2\alpha s_x - u s_x - u_x s. \quad (21)$$

Equation (19) includes the KdV, CH, and Hunter-Saxton (HS) [29] equations. The KdV equation can be obtained by putting $c_1 = c_4 = 0$. The HS equation

$$u_{tx} + \frac{1}{2}u_x^2 + uu_{xx} = 0 \quad (22)$$

can be obtained by putting $c_1 = c_2 = c_3 = 0$ and integrating with respect to x . The BT in this case is

$$x \rightarrow x - \frac{2\alpha}{s}, \quad u \rightarrow u - \frac{2\alpha u_x}{s} - 2\alpha,$$

where

$$s_x = -\frac{s^2}{2\alpha} - \frac{u_{xx}}{2}, \quad (23)$$

$$s_t = \alpha u_{xx} + 2\alpha s_x - u s_x - u_x s. \quad (24)$$

4 The double Bäcklund transformation and superposition principles

In this section we discuss double BTs for CH and pCH. We also show the superposition principles for these equations.

As we saw in the previous section, a BT (which acts on the CH fields $u, m, p = \sqrt{m}$, the pCH fields v, n and the independent coordinate x according to equations (10), (16), (15), (18), (17), (14) respectively) is determined by a solution s of (11),(12). We use

the following notation: Denote by s_α, s_β etc the solutions of (11),(12) corresponding to parameters α, β etc. Denote the associated action on the fields by $u \rightarrow u_\alpha, m \rightarrow m_\alpha$ etc. Denote by $s_{\alpha\beta}$ the solution of (11),(12) with u, m replaced by u_α, m_α and parameter β (i.e. we start with a solution obtained from a BT with parameter α and are now considering acting upon it by a further BT with parameter β). Denote the corresponding action on the fields by $u_\alpha \rightarrow u_{\alpha\beta}, m_\alpha \rightarrow m_{\alpha\beta}$ etc.

The fundamental fact about double BTs, as proved in [60], is that they commute, i.e. $u_{\alpha\beta} = u_{\beta\alpha}, m_{\alpha\beta} = m_{\beta\alpha}$ etc. From, for example, the transformation law for the pCH field n , (17), it immediately follows that

$$s_\alpha + s_{\alpha\beta} = s_\beta + s_{\beta\alpha} . \quad (25)$$

Checking the consistency of this with the versions of (11) and (12) satisfied by $s_\alpha, s_\beta, s_{\alpha\beta}, s_{\beta\alpha}$ we obtain

$$s_{\alpha\beta} = -s_\alpha + \frac{(\alpha - \beta)(\alpha\beta - s_\alpha s_\beta)}{\beta s_\alpha - \alpha s_\beta} , \quad s_{\beta\alpha} = -s_\beta + \frac{(\alpha - \beta)(\alpha\beta - s_\alpha s_\beta)}{\beta s_\alpha - \alpha s_\beta} . \quad (26)$$

In fact it is possible to check directly that these formulas for $s_{\alpha\beta}, s_{\beta\alpha}$ give solutions of the relevant versions of (11) and (12) without any need to assume (25).

From (26) it follows that once s_α and s_β are known, it is possible to immediately find the action of a double BT. Using the transformation laws for m, p, n, x and (26) we find

$$p_{\alpha\beta} = \frac{\alpha\beta((s_\alpha - s_\beta)^2 - (\alpha - \beta)^2)}{(\beta s_\alpha - \alpha s_\beta)^2} p \quad (27)$$

$$m_{\alpha\beta} = \frac{\alpha^2\beta^2((s_\alpha - s_\beta)^2 - (\alpha - \beta)^2)^2}{(\beta s_\alpha - \alpha s_\beta)^4} m \quad (28)$$

$$n_{\alpha\beta} = n - 2 \frac{(\alpha - \beta)(\alpha\beta - s_\alpha s_\beta)}{\beta s_\alpha - \alpha s_\beta} \quad (29)$$

$$x_{\alpha\beta} = x + \ln \left| \frac{s_\beta - s_\alpha + \alpha - \beta}{s_\beta - s_\alpha - \alpha + \beta} \right| \quad (30)$$

For u and v we proceed as follows. From (10) and (18) we obtain

$$u_\alpha + u + \frac{1}{\alpha}(v_\alpha - v)s_\alpha = -2\alpha \quad (31)$$

and similarly

$$u_\beta + u + \frac{1}{\beta}(v_\beta - v)s_\beta = -2\beta , \quad (32)$$

$$u_{\alpha\beta} + u_\beta + \frac{1}{\alpha}(v_{\alpha\beta} - v_\beta)s_{\beta\alpha} = -2\alpha , \quad (33)$$

$$u_{\alpha\beta} + u_\alpha + \frac{1}{\beta}(v_{\alpha\beta} - v_\alpha)s_{\alpha\beta} = -2\beta . \quad (34)$$

Eliminating $v_\alpha, v_\beta, v_{\alpha\beta}$ from these 4 relations, using (26) for $s_{\alpha\beta}$ and (10) for u_α, u_β we obtain

$$u_{\alpha\beta} = u - \frac{2(\alpha - \beta)((\alpha - \beta)(\alpha + \beta + u) + (s_\beta - s_\alpha)(s_\beta + s_\alpha + u_x))}{(\alpha - \beta)^2 - (s_\beta - s_\alpha)^2} . \quad (35)$$

Similarly, by first eliminating the fields u ,

$$v_{\alpha\beta} = v - \frac{2(\alpha - \beta)((\alpha - \beta)u_x + (s_\beta - s_\alpha)u + 2(\alpha s_\beta - \beta s_\alpha))}{(\alpha - \beta)^2 - (s_\beta - s_\alpha)^2}. \quad (36)$$

Equations (27),(28),(29),(30),(35) and (36) are algebraic formulas for the implementation of a double BT given s_α and s_β . However s_α and s_β also determine the implementation of the original single BTs, so it is natural to try to eliminate them to obtain nonlinear superposition formulae for each of the quantities p, m, n, x, u, v . For example, for x we have, from (14),

$$s_\alpha = \frac{e^x + e^{x_\alpha}}{e^x - e^{x_\alpha}} \alpha, \quad s_\beta = \frac{e^x + e^{x_\beta}}{e^x - e^{x_\beta}} \beta$$

and using these in (30) gives

$$\frac{(e^x - e^{x_\alpha})(e^{x_\beta} - e^{x_{\alpha\beta}})}{(e^x - e^{x_\beta})(e^{x_\alpha} - e^{x_{\alpha\beta}})} = \frac{\alpha}{\beta}. \quad (37)$$

Thus we see e^x satisfies the cross-ratio equation, equation A1[$\delta = 0$] in the ABS classification [1]. Similarly for n we obtain

$$\beta(2\alpha + n - n_\alpha)(2\alpha - n_\beta + n_{\beta\alpha}) = \alpha(2\beta + n - n_\beta)(2\beta - n_\alpha + n_{\alpha\beta}), \quad (38)$$

which is also the cross-ratio equation after a simple field redefinition. For p the situation is a little more complicated as we have

$$s_\alpha^2 = \alpha(\alpha + pp_\alpha), \quad s_\beta^2 = \beta(\beta + pp_\beta),$$

and knowledge of p_α only determines s_α up to a sign. As a result, for given p, p_α, p_β there are 4 possibilities for $p_{\alpha\beta}$, which are given by solutions of the two multiquadratic quad-graph equations

$$4\alpha\beta(\alpha - \beta)(p - p_{\alpha\beta})(p_\alpha - p_\beta) + \alpha\beta(p - p_{\alpha\beta})^2(p_\alpha - p_\beta)^2 + \alpha(\alpha - \beta)(pp_\alpha - p_\beta p_{\alpha\beta})^2 + \beta(\beta - \alpha)(pp_\beta - p_\alpha p_{\alpha\beta})^2 = 0, \quad (39)$$

$$-4\alpha\beta(\alpha + \beta)(p + p_{\alpha\beta})(p_\alpha + p_\beta) - \alpha\beta(p + p_{\alpha\beta})^2(p_\alpha + p_\beta)^2 + \alpha(\alpha + \beta)(pp_\alpha - p_\beta p_{\alpha\beta})^2 + \beta(\beta + \alpha)(pp_\beta - p_\alpha p_{\alpha\beta})^2 = 0. \quad (40)$$

The first of these is precisely the H3* equation in the Atkinson-Nieszporski classification of integrable multiquadratic quad graph equations [2], as is the second after a simple field redefinition.

For u and v we have not succeeded to write a single superposition principle not involving any of the other fields. However, using the relations (31)-(34) it is possible to write the following superposition principles involving, respectively, just u and n , and just v and n :

$$\alpha \left(\frac{u_\alpha + u + 2\alpha}{n_\alpha - n} - \frac{u_{\alpha\beta} + u_\beta + 2\alpha}{n_{\alpha\beta} - n_\beta} \right) - \beta \left(\frac{u_\beta + u + 2\beta}{n_\beta - n} - \frac{u_{\alpha\beta} + u_\alpha + 2\beta}{n_{\alpha\beta} - n_\alpha} \right) = 0 \quad (41)$$

$$\begin{aligned} & \frac{(v_\beta - v)(n_\beta - n) + (v_{\alpha\beta} - v_\alpha)(n_{\alpha\beta} - n_\alpha)}{\beta} - \frac{(v_\alpha - v)(n_\alpha - n) + (v_{\alpha\beta} - v_\beta)(n_{\alpha\beta} - n_\beta)}{\alpha} \\ &= 8(\beta - \alpha). \end{aligned} \quad (42)$$

Here the fields n satisfy the cross-ratio type equation (38).

5 Travelling wave solutions

In this section we apply the BT (10),(14) where s satisfies (11),(12) to the constant solution of CH $u = u_0 \neq 0$, to obtain travelling wave solutions, specifically soliton and cuspon solutions. These and other travelling wave solutions have been extensively studied in the literature, see for example [40, 41, 7, 8, 33, 47, 48, 50, 37, 54], and the BT is just one of many methods to derive them. The advantages of the BT will become apparent when we study superposition in the next section.

If $\alpha(\alpha + u_0) > 0$ there are two kinds of real solutions of (11),(12):

$$s_\alpha = \sqrt{\alpha(u_0 + \alpha)} \tanh \left(\frac{\sqrt{\alpha(u_0 + \alpha)}(x - x_0 + (2\alpha - u_0)t)}{2\alpha} \right), \quad (43)$$

which we call the “tanh-type” solution, and the same with tanh replaced by coth, which we call the “coth-type” solution. As we will see both of these give rise to travelling wave solutions. If $\alpha(\alpha + u_0) < 0$ then there are real solutions

$$s_\alpha = \sqrt{-\alpha(u_0 + \alpha)} \tan \left(\frac{\sqrt{-\alpha(u_0 + \alpha)}(x - x_0 + (2\alpha - u_0)t)}{2\alpha} \right), \quad (44)$$

and the same with tan replaced by cot, and an overall minus sign. Both of these give rise to periodic solutions (see for example [7, 37]), but these will not be studied here.

Returning to the case $\alpha(\alpha + u_0) > 0$, it is useful to write $\alpha + u_0 = \alpha U^2$, where $U > 0$, so the solution (43) becomes

$$s_\alpha = \alpha U \tanh \left(\frac{U}{2} (x - x_0 + (3 - U^2)\alpha t) \right), \quad (45)$$

and the same with coth for a coth-type solution. Using (10),(14) the resulting solution is $u_\alpha(x_\alpha, t)$ where

$$u_\alpha = \alpha(U^2 - 3) + \frac{2\alpha(U^2 - 1)}{U^2 \tanh^2 \left(\frac{U}{2} (x - x_0 + (3 - U^2)\alpha t) \right) - 1}, \quad (46)$$

$$x_\alpha = x + \ln \left| \frac{U \tanh \left(\frac{U}{2} (x - x_0 + (3 - U^2)\alpha t) \right) - 1}{U \tanh \left(\frac{U}{2} (x - x_0 + (3 - U^2)\alpha t) \right) + 1} \right| \quad (47)$$

or the same with coth. Finally, writing $z = x - x_0 + (3 - U^2)\alpha t$, the solution becomes $u_\alpha(x_\alpha, t)$ where

$$u_\alpha = \alpha(U^2 - 3) + \frac{2\alpha(U^2 - 1)}{U^2 \tanh^2 \frac{1}{2} U z - 1}, \quad (48)$$

$$x_\alpha - x_0 + (3 - U^2)\alpha t = z + \ln \left| \frac{U \tanh \frac{1}{2} U z - 1}{U \tanh \frac{1}{2} U z + 1} \right|, \quad (49)$$

this being a tanh-type solution, or a coth-type solution, which is the same with tanh replaced by coth. Both tanh-type and coth-type solutions are travelling waves with speed $c = (U^2 - 3)\alpha$, written in an implicit form. The first step in analyzing these solutions is to decide whether the maps from x_α to z are bijections. For tanh-type solutions with $U < 1$, neither the factor in the numerator or in the denominator inside the ln can vanish, and

thus x_α only tends to (plus or minus) infinity as z tends to (plus or minus) infinity. The corresponding solutions are solitons which tend to $u_0 = \alpha(U^2 - 1)$ at spatial infinity, with speed $c = \alpha(U^2 - 3)$ and central elevation $-\alpha(1 + U^2) = c - 2u_0$. Note that since

$$U = \sqrt{\frac{3u_0 - c}{u_0 - c}}$$

and $0 < U < 1$ we must either have $c < 3u_0 < 0$ or $0 < 3u_0 < c$. Figure 1 displays the soliton profile for $c = 2$ and $u_0 = 0.5, 0.1, 0.02$. (For negative u_0 and c the soliton is inverted.)

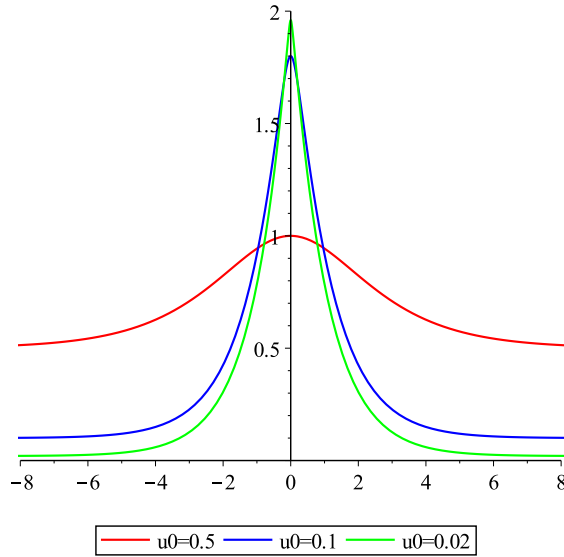


Figure 1: Soliton profile, $c = 2$, $u_0 = 0.5, 0.1, 0.02$.

Of particular interest is the limit of the soliton for fixed c and $u_0 \downarrow 0$ (for $c > 0$) or $u_0 \uparrow 0$ (for $c < 0$). Figure 2 shows x_α as a function of z (for $x_0 = t = 0$), u_α as a function of z and u_α as a function of x_α in the case $c = 2$, $u_0 = 10^{-8}$. x_α is close to zero, and u_α is close to c for a large interval of z values of size $O(|\ln(u_0/c)|)$ around $z = 0$. In the plot of u_α against x_α this gives rise to a sharp peak. This is the peakon limit. To see this analytically it is possible to use (48) to find z in terms of u (with a \pm uncertainty as it is necessary to take a square root), and then (49) becomes

$$\begin{aligned} x_\alpha - x_0 + (3 - U^2)\alpha t &= \pm \left(2\sqrt{\frac{c - u_0}{c - 3u_0}} \operatorname{arctanh} \left(\sqrt{\frac{c - u_0}{c - 3u_0}} \sqrt{1 - \frac{2u_0}{c - u_\alpha}} \right) \right. \\ &\quad \left. + \ln \left(\frac{c - u_\alpha}{2u_0} \left(1 - \sqrt{1 - \frac{2u_0}{c - u_\alpha}} \right)^2 \right) \right). \end{aligned} \quad (50)$$

Both terms on the RHS diverge as $u_0 \rightarrow 0$, but it is straightforward to extract the divergent behavior, which cancels between the terms, and to obtain the limit, which is simply $\pm \ln \left(\frac{u_\alpha}{c} \right)$.

Moving now to tanh-type solutions with $U > 1$, from (49) we expect x_α to diverge when $\tanh \frac{1}{2}Uz = \pm \frac{1}{U}$ and thus the map from z to x_α will not be a bijection. Figure 3 shows x_α and u_α as functions of z for $c = 2$ and $u_0 = 3$. The map from z to x_α is 3 to

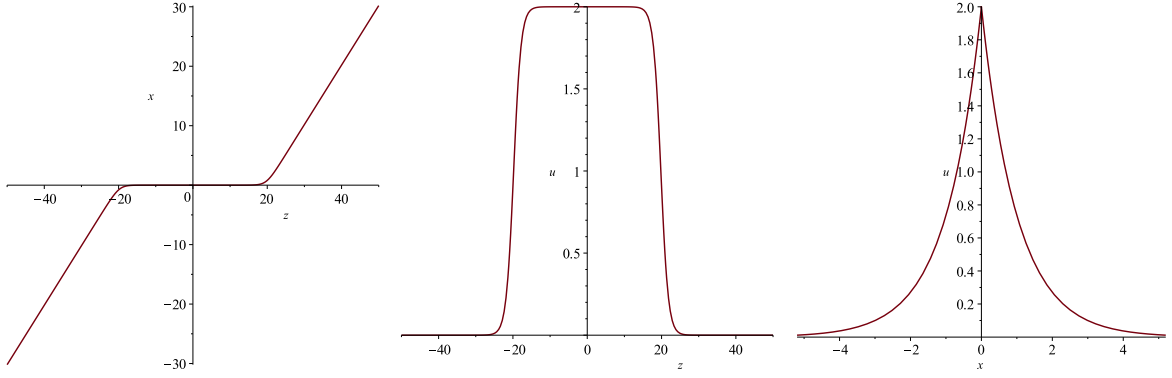


Figure 2: The soliton with $c = 2$, approaching the peakon limit. $u_0 = 10^{-8}$. x as a function of z , u as a function of z and u as a function of x .

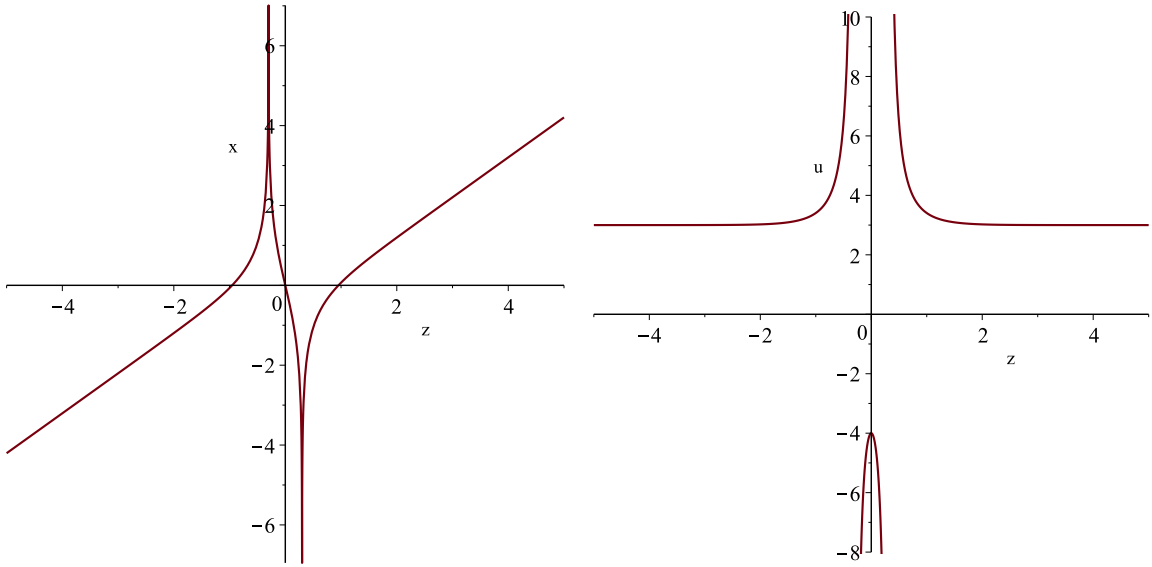


Figure 3: tanh-type solutions with $c = 2$ and $u_0 = 3$ ($U = \sqrt{7} > 1$), x_α and u_α as functions of z . The map from z to x_α is not $1 - 1$.

1 and thus there are 3 corresponding solutions of CH, depicted in Figure 4. Since these are all unbounded we do not devote further attention to them.

Moving now to coth-type solutions, the situation is very similar, but now the map from z to x_α will be $1 - 1$ if $U > 1$ and many to 1 if $U < 1$, and there is a subtlety arising due to the divergence of $\coth \frac{1}{2}Uz$ at $z = 0$. For coth-type solutions with $U < 1$, x_α diverges when $\coth \frac{1}{2}Uz = \pm \frac{1}{U}$. The map from z to x_α is once again 3 to 1. Figure 5 shows x_α and u_α as functions of z for $c = 2$ and $u_0 = \frac{1}{2}$, and Figure 6 shows the 3 corresponding solutions of CH. The subtlety, as can be seen in Figure 7, is that the solution corresponding to the range of z 's that includes zero, has a cusp at $z = 0$, arising from the divergence of $\coth \frac{1}{2}Uz$. Since at this point u_α is not differentiable, it is necessary to ask in what sense this is a solution of CH. Fortunately, the value of u_α at the cusp is c , which makes it possible to interpret the solution in a weak sense [37], though we do not go into details here.

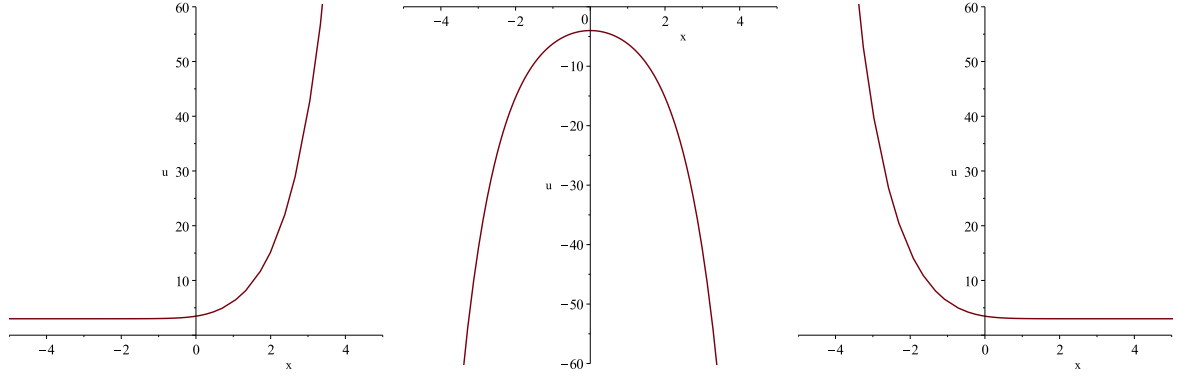


Figure 4: tanh-type solutions with $c = 2$ and $u_o = 3$ ($U = \sqrt{7} > 1$), u_α as a function of x_α for the 3 unbounded solutions of CH.

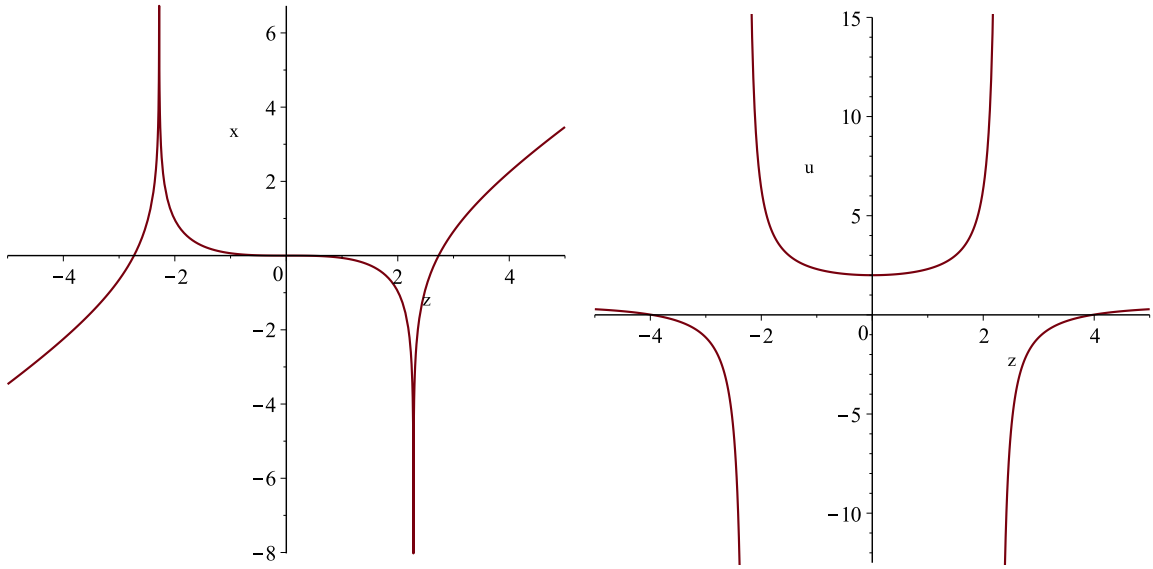


Figure 5: coth-type solutions with $c = 2$ and $u_0 = \frac{1}{2}$ ($U = \frac{1}{\sqrt{3}} < 1$), x_α and u_α as functions of z . The map from z to x_α is not 1-1.

For coth-type solutions with $U > 1$, the map from z to x_α is a bijection, and once again there is a single solution of CH, but with a cusp at $z = 0$ — this is the cuspon solution. Due to the requirement $U > 1$ cuspon solutions only exist with speed $c < u_0$ if u_0 is positive, and speed $c > u_0$ if u_0 is negative. Figure 8 illustrates cuspon solutions with $c = 2$ for $u_0 = -1, -0.5, -0.1$. (For positive u_0 the cuspon is inverted.) Note that the central elevation of the cusp is c , as required for it to be a weak solution. For $c > 0$ ($c < 0$) it is possible to consider the limit of the cuspon as $u_0 \uparrow 0$ ($u_0 \downarrow 0$), and this is once again the peakon limit.

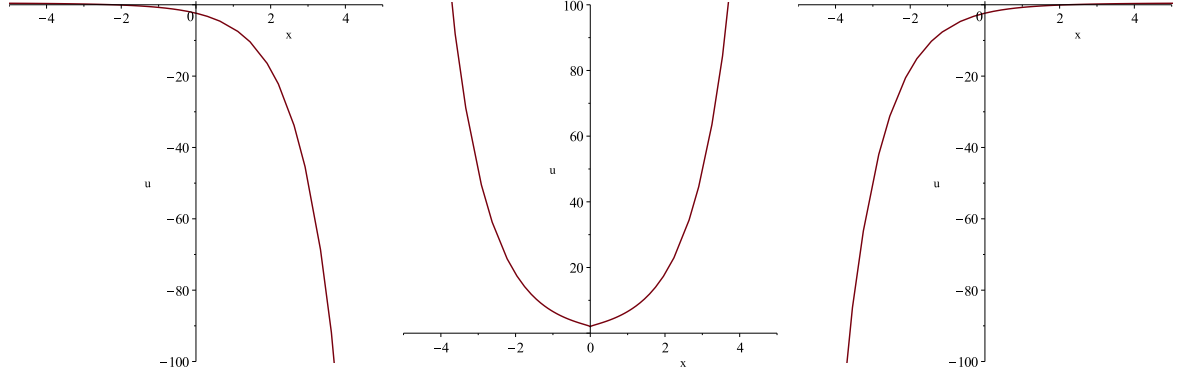


Figure 6: coth-type solutions with $c = 2$ and $u_0 = \frac{1}{2}$ ($U = \frac{1}{\sqrt{3}} < 1$), u_α as a function of x_α for the 3 unbounded solutions of CH.

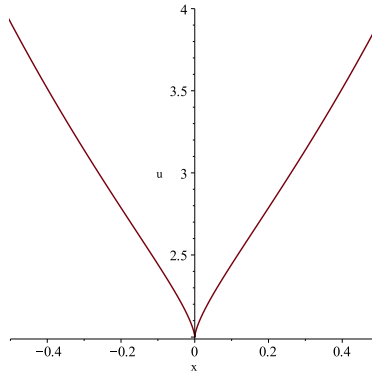


Figure 7: coth-type solutions with $c = 2$ and $u_0 = \frac{1}{2}$ ($U = \frac{1}{\sqrt{3}} < 1$), close up on the cusp in one of the solutions of CH.

We summarize the travelling waves presented in this section in the following table. All the solutions have asymptotic height u_0 :

tanh-type	$U < 1$	soliton	central elevation $c - 2u_0$ $u_0 > 0$: $c > 3u_0$ $u_0 < 0$: $c < 3u_0$ inverted
	$U > 1$	unphysical	
coth-type	$U < 1$	unphysical	
	$U < 1$	cuspon	central elevation c $u_0 > 0$: $c < u_0$ inverted $u_0 < 0$: $c > u_0$

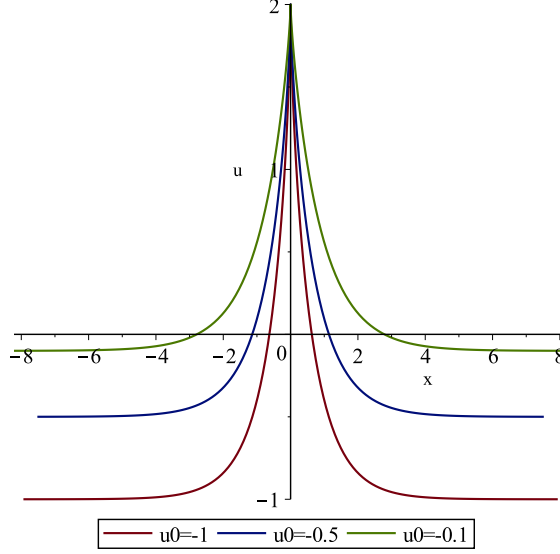


Figure 8: Cuspon profile, $c = 2$, $u_0 = -1, -0.5, -0.1$.

6 Two wave solutions

The first investigations of two wave solutions were [60] and [22], both of which required some element of numerical computation. However, since then, a substantial literature [31, 39, 49, 18, 38, 43, 53, 44, 52, 19, 63] has developed on multisoliton, multicuspon and soliton-cuspon solutions. The known methods for analytic construction of solutions include a determinantal formula based on the inverse scattering approach, a Hirota bilinear form for CH and a reciprocal transformation relating the CH hierarchy to the KdV hierarchy. (For multipeakon solutions very different techniques are involved [9, 4, 6, 5, 51].) As we will shortly see, use of the superposition principle gives a further very simple method.

In our approach, two wave solutions should be obtained using formulas (35) and (30), taking $u = u_0$ to be constant and s_α (s_β) either of tanh-type, as given in (45) where $U = U_\alpha = \sqrt{1 + \frac{u_0}{\alpha}}$ ($U = U_\beta = \sqrt{1 + \frac{u_0}{\beta}}$) and $x_0 = x_{\alpha,0}$ ($x_0 = x_{\beta,0}$), or of coth-type, which is identical but with coth. The only question is which superpositions of this type give maps from x to $x_{\alpha\beta}$ that are 1-1.

Proposition. The following 3 superpositions give maps from x to $x_{\alpha\beta}$ which are 1-1:

1. tanh-type solutions s_α with $U_\alpha < 1$ with tanh-type solutions s_β with $U_\beta > 1$ (so $\frac{u_0}{\alpha} < 0 < \frac{u_0}{\beta}$) — soliton-cuspon superpositions.
2. tanh-type solutions s_α with $U_\alpha < 1$ with coth-type solutions s_β with $U_\beta < 1$, with $U_\alpha < U_\beta$ — soliton-soliton superpositions.
3. tanh-type solutions s_α with $U_\alpha > 1$ with coth-type solutions s_β with $U_\beta > 1$, with $U_\beta < U_\alpha$ — cuspon-cuspon superpositions.

Note here, for example, that a soliton-soliton superposition is *not* as we might expect, the superposition of two tanh-type solutions with $U < 1$, but the superposition of a tanh-type solution with $U < 1$ with a unphysical coth-type solution with $U < 1$.

Proof. It is necessary to show in each case that neither the numerator or denominator of the expression inside the \ln in (30) vanishes, i.e. that $|s_\beta - s_\alpha| \neq |\beta - \alpha|$. In the calculations below we repeatedly use the identities

$$\alpha = \frac{u_0}{U_\alpha^2 - 1}, \quad \beta = \frac{u_0}{U_\beta^2 - 1}.$$

1. In this case we have

$$\begin{aligned} |s_\beta - s_\alpha| &= |\beta U_\beta \tanh(\dots) - \alpha U_\alpha \tanh(\dots)| \\ &< |\beta U_\beta| + |\alpha U_\alpha| \quad \text{as } |\tanh| < 1 \\ &= \left| \frac{u_0 U_\beta}{U_\beta^2 - 1} \right| + \left| \frac{u_0 U_\alpha}{U_\alpha^2 - 1} \right| \\ &= |u_0| \left(\frac{U_\beta}{U_\beta^2 - 1} - \frac{U_\alpha}{U_\alpha^2 - 1} \right) \quad \text{as } 0 < U_\alpha < 1 < U_\beta \\ &= |u_0| \left(\frac{1}{U_\beta^2 - 1} + \frac{1}{U_\beta + 1} - \frac{1}{U_\alpha^2 - 1} - \frac{1}{U_\alpha + 1} \right) \\ &< |u_0| \left(\frac{1}{U_\beta^2 - 1} - \frac{1}{U_\alpha^2 - 1} \right) \quad \text{as } \frac{1}{U_\alpha + 1} > \frac{1}{U_\beta + 1} \\ &= |\beta - \alpha|. \end{aligned}$$

2. In this case we have

$$\begin{aligned} |s_\alpha| + |\beta - \alpha| &= |\alpha U_\alpha \tanh(\dots)| + |\beta - \alpha| \\ &< |\alpha U_\alpha| + |\beta - \alpha| \quad \text{as } |\tanh| < 1 \\ &= \left| \frac{u_0 U_\alpha}{U_\alpha^2 - 1} \right| + \left| \frac{u_0}{U_\beta^2 - 1} - \frac{u_0}{U_\alpha^2 - 1} \right| \\ &= |u_0| \left(\frac{U_\alpha}{1 - U_\alpha^2} + \frac{1}{1 - U_\beta^2} - \frac{1}{1 - U_\alpha^2} \right) \quad \text{as } 0 < U_\alpha < U_\beta < 1 \\ &= |u_0| \left(\frac{1}{1 - U_\beta^2} - \frac{1}{1 + U_\alpha} \right) \\ &= |u_0| \left(\frac{U_\beta}{1 - U_\beta^2} + \frac{1}{1 + U_\beta} - \frac{1}{1 + U_\alpha} \right) \\ &< |u_0| \frac{U_\beta}{1 - U_\beta^2} \quad \text{as } \frac{1}{1 + U_\beta} < \frac{1}{1 + U_\alpha} \\ &= |\beta U_\beta| \\ &< |\beta U_\beta \coth(\dots)| \quad \text{as } |\coth| > 1 \\ &= |s_\beta|. \end{aligned}$$

This contradicts $|s_\beta - s_\alpha| = |\beta - \alpha|$, as the latter implies $|s_\beta| \leq |s_\alpha| + |\beta - \alpha|$.

3. Similarly to case 2 we have

$$\begin{aligned}
|s_\alpha| + |\beta - \alpha| &= |\alpha U_\alpha \tanh(\dots)| + |\beta - \alpha| \\
&< |\alpha U_\alpha| + |\beta - \alpha| \quad \text{as } |\tanh| < 1 \\
&= \left| \frac{u_0 U_\alpha}{U_\alpha^2 - 1} \right| + \left| \frac{u_0}{U_\beta^2 - 1} - \frac{u_0}{U_\alpha^2 - 1} \right| \\
&= |u_0| \left(\frac{U_\alpha}{U_\alpha^2 - 1} + \frac{1}{U_\beta^2 - 1} - \frac{1}{U_\alpha^2 - 1} \right) \quad \text{as } 1 < U_\beta < U_\alpha \\
&= |u_0| \left(\frac{1}{U_\beta^2 - 1} + \frac{1}{U_\alpha + 1} \right) \\
&= |u_0| \left(\frac{U_\beta}{U_\beta^2 - 1} - \frac{1}{U_\beta + 1} + \frac{1}{U_\alpha + 1} \right) \\
&< |u_0| \frac{U_\beta}{U_\beta^2 - 1} \quad \text{as } \frac{1}{U_\alpha + 1} < \frac{1}{U_\beta + 1} \\
&= |\beta U_\beta| \\
&< |\beta U_\beta \coth(\dots)| \quad \text{as } |\coth| > 1 \\
&= |s_\beta|.
\end{aligned}$$

It remains to present the plots of some superpositions. Figure 9 shows a tanh-tanh superposition with $u_0 = 1$, $x_{\alpha,0} = 0$, $x_{\beta,0} = 2$, $c_\alpha = u_0 - 2\alpha = 4$, $c_\beta = u_0 - 2\beta = -1$. (For $u_0 > 0$ such solutions exist provided $c_\alpha > 3u_0$ and $c_\beta < u_0$ — note here that c_β can be positive or negative.) Figure 10 shows a tanh-coth soliton-soliton superposition with $u_0 = 1$, $x_{\alpha,0} = 0$, $x_{\beta,0} = -10$, $c_\alpha = u_0 - 2\alpha = 4$, $c_\beta = u_0 - 2\beta = 6$. Figure 11 shows a tanh-coth cuspon-cuspon superposition with $u_0 = 1$, $x_{\alpha,0} = 0$, $x_{\beta,0} = -2$, $c_\alpha = u_0 - 2\alpha = -1$, $c_\beta = u_0 - 2\beta = -2$.

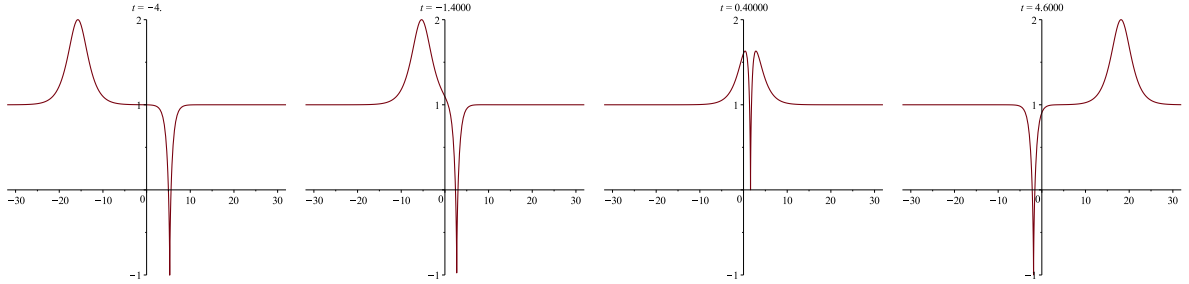


Figure 9: Tanh-Tanh Superposition. $u_0 = 1$, $x_{\alpha,0} = 0$, $x_{\beta,0} = 2$, $c_\alpha = 4$, $c_\beta = -1$. Plots for times $t = -4, -1.4, 0.4, 4.6$ from left to right.

Note that in all the plots we have taken $u_0 > 0$, in which case, as in the previous section, the soliton solutions have positive speed $c > 3u_0$ and central elevation $c - 2u_0 > u_0$, whereas the cuspon solutions have speed $c < u_0$, which can be positive or negative, and central elevation $c < u_0$ (i.e. they might be called “anticuspons”). In the case $u_0 < 0$ everything is inverted and reversed (reflecting the $t \rightarrow -t$, $u \rightarrow -u$ symmetry of (2): solitons have negative speed $c < 3u_0$ and central elevation $c - 2u_0 < u_0$ (antisolitons), and cuspons have speed $c > u_0$ and central elevation $c > u_0$.

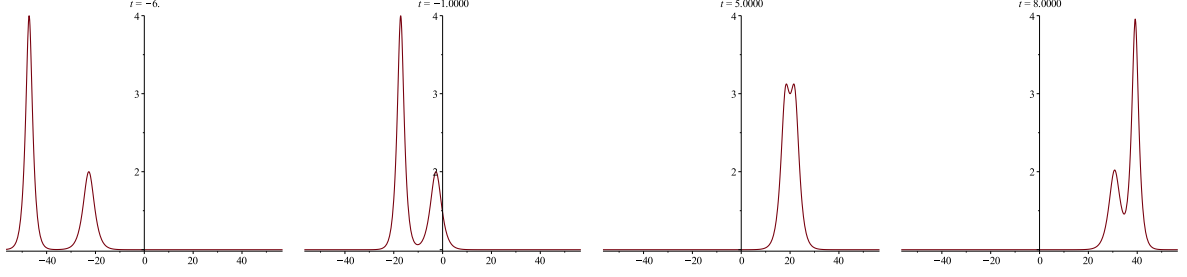


Figure 10: Tanh-Coth Soliton-Soliton Superposition. $u_0 = 1$, $x_{\alpha,0} = 0$, $x_{\beta,0} = -10$, $c_\alpha = 4$, $c_\beta = 6$. Plots for times $t = -6, -1, 5, 8$ from left to right.

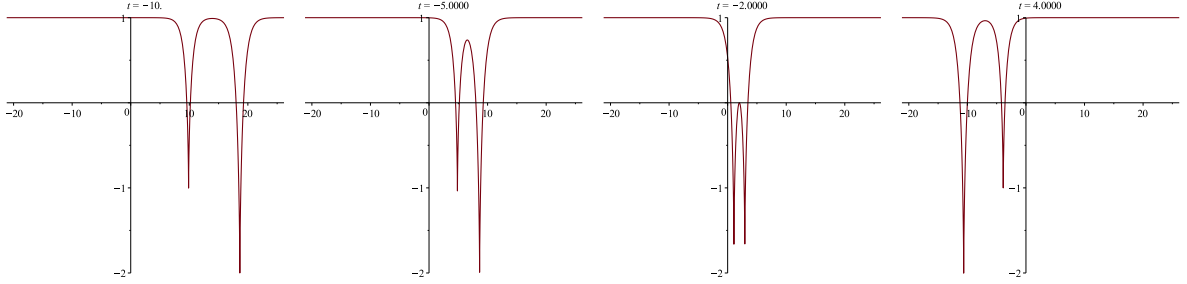


Figure 11: Tanh-Coth Cuspon-Cuspon Superposition. $u_0 = 1$, $x_{\alpha,0} = 0$, $x_{\beta,0} = -2$, $c_\alpha = -1$, $c_\beta = -2$. Plots for times $t = -10, -5, -2, 4$ from left to right.

7 Symmetries and conservation laws for the Camassa-Holm equation

In this section we show how to use the BT to obtain infinite hierarchies of symmetries and conservation laws for CH and pCH, following the general methodology described in [55]. The discussion of symmetries of CH in the literature is limited, though the existence of an infinite number of symmetries is implicit from the bihamiltonian structure given in [9]. In the series of papers [56, 57, 28, 25, 27], Reyes and collaborators present *nonlocal* symmetries of CH depending on a parameter, and then expand in powers of the parameter to obtain local symmetries, though limited details are given. Some explicit formulae appear in [24]. Our approach is related, but we will not discuss the connection explicitly.

As a starting point for our discussion of symmetries we could take the generating symmetry (9) for aCH, and work out the induced action on x , the independent variable in CH. But a more direct approach is to look at the superposition principle (35),(30) in the limit that β tends to α , but s_β tends to a second solution of (11)-(12) *distinct from* s_α . More explicitly, setting $\beta = \alpha - \frac{1}{2}\epsilon$, $s_\alpha = s_\alpha^{(1)}$, $s_\beta = s_\alpha^{(2)} + O(\epsilon)$ in (35),(30) we obtain

$$\begin{aligned} u_{\alpha, \alpha - \frac{1}{2}\epsilon} &= u + \epsilon \frac{s_\alpha^{(1)} + s_\alpha^{(2)} + u_x}{s_\alpha^{(2)} - s_\alpha^{(1)}} + O(\epsilon^2), \\ x_{\alpha, \alpha - \frac{1}{2}\epsilon} &= x + \frac{\epsilon}{s_\alpha^{(2)} - s_\alpha^{(1)}} + O(\epsilon^2). \end{aligned}$$

We deduce the generating symmetry for CH $X = Q^x \frac{\partial}{\partial x} + Q^u \frac{\partial}{\partial u}$ where

$$Q^x = \frac{1}{s_\alpha^{(2)} - s_\alpha^{(1)}}, \quad Q^u = \frac{s_\alpha^{(1)} + s_\alpha^{(2)} + u_x}{s_\alpha^{(2)} - s_\alpha^{(1)}}. \quad (51)$$

Here $s_\alpha^{(1)}, s_\alpha^{(2)}$ are two different solutions of (11),(12) for the same parameter α . This symmetry depends upon α ; expansion in a (formal) power series in α will give an infinite hierarchy of symmetries. However before we do this, we exploit the fact that a generalized symmetry of the form $X = Q^x \frac{\partial}{\partial x} + Q^u \frac{\partial}{\partial u}$ which acts on both the dependent and independent variables can be transformed to a generalized symmetry which acts only on the dependent variable [46] with characteristic $Q = Q^u - Q^x u_x$. Here we have

$$Q = \frac{s_\alpha^{(1)} + s_\alpha^{(2)}}{s_\alpha^{(2)} - s_\alpha^{(1)}}. \quad (52)$$

This is also the characteristic for a symmetry of the full family of equations (19).

The next thing to do is to find a (formal) asymptotic series solution of (11)-(12) for small $|\alpha|$. This takes the form

$$s_\alpha = \sum_{n=1}^{\infty} s_n \alpha^{\frac{n}{2}}, \quad (53)$$

where

$$s_1 = \sqrt{m}, \quad s_2 = -\frac{s_{1,x}}{s_1}, \quad s_{n+1} = -\frac{s_{n,x}}{s_1} + \frac{1}{2s_1} \left(\delta_{n,2} - \sum_{i=0}^{n-2} s_{i+2} s_{n-i} \right), \quad n = 2, 3, \dots$$

A second solution of (11)-(12) can be obtained by replacing $\alpha^{\frac{1}{2}}$ by $-\alpha^{\frac{1}{2}}$. So we get

$$s_\alpha^{(1)} = \sum_{n=1}^{\infty} s_n \alpha^{\frac{n}{2}}, \quad s_\alpha^{(2)} = \sum_{n=1}^{\infty} s_n (-\alpha^{\frac{1}{2}})^n. \quad (54)$$

Plugging this into (52) we obtain

$$\frac{Q}{\sqrt{\alpha}} = \frac{\sum_{n=1}^{\infty} s_{2n} \alpha^n}{\sum_{n=1}^{\infty} s_{2n-1} \alpha^n}. \quad (55)$$

The expansion of (55) around $\alpha = 0$ gives an infinite hierarchy of symmetries of CH. The first few of these take the form

$$X_1 = \left(\frac{1}{\sqrt{m}} \right)_x \frac{\partial}{\partial u}, \quad (56)$$

$$X_2 = \left(\frac{4mm_{xx} - 5m_x^2 + 4m^2}{m^{7/2}} \right)_x \frac{\partial}{\partial u}, \quad (57)$$

$$X_3 = \left(\frac{64m^3 m_{xxxx} - 448m^2 m_x m_{xxx} - 160m^3 m_{xx} + 1848mm_x^2 m_{xx} - 336m^2 m_{xx}^2 + 280m^2 m_x^2 - 1155m_x^4 - 48m^4}{m^{13/2}} \right)_x \frac{\partial}{\partial u}. \quad (58)$$

The fact that all the characteristics are x -derivatives is indicative that these symmetries can be derived from corresponding symmetries of pCH. The generating symmetry for pCH (up to an irrelevant overall constant factor) is thus $Q^v \frac{\partial}{\partial v}$ where

$$Q^v = \frac{1}{s_\alpha^{(1)} - s_\alpha^{(2)}}. \quad (59)$$

As stressed before, the symmetry with characteristic (52) is a symmetry for the full family of equations (19), including the HS equation. For HS the asymptotic series solutions of (23)-(24) takes the form

$$s_\alpha^{(1)} = \sum_{n=1}^{\infty} s_n \alpha^{\frac{n}{2}}, \quad s_\alpha^{(2)} = \sum_{n=1}^{\infty} s_n (-\alpha^{\frac{1}{2}})^n, \quad (60)$$

where

$$s_1 = i\sqrt{u_{xx}}, \quad s_2 = -\frac{s_{1,x}}{s_1}, \quad s_{n+1} = -\frac{s_{n,x}}{s_1} - \frac{1}{2s_1} \left(\sum_{i=0}^{n-2} s_{i+2} s_{n-i} \right), \quad n = 2, 3, \dots$$

Proceeding as before gives an infinite hierarchy of symmetries for HS, with the first few taking the form

$$\begin{aligned} X_1 &= \left(\frac{1}{\sqrt{u_{xx}}} \right)_x \frac{\partial}{\partial u}, \\ X_2 &= \left(4 \frac{u_{xxxx}}{u_{xx}^{5/2}} - 5 \frac{u_{xxx}^2}{u_{xx}^{7/2}} \right)_x \frac{\partial}{\partial u}, \\ X_3 &= \left(1155 \frac{u_{xxx}^4}{u_{xx}^{13/2}} - 1848 \frac{u_{xxx}^2 u_{xxxx}}{u_{xx}^{11/2}} + 448 \frac{u_{xxx} u_{xxxxx}}{u_{xx}^{9/2}} + 336 \frac{u_{xxx}^2}{u_{xx}^{9/2}} - 64 \frac{u_{xxxxxx}}{u_{xx}^{7/2}} \right)_x \frac{\partial}{\partial u}. \end{aligned}$$

Using the fact that if a single solution of the Riccati equation (11) is known then it is possible to find the general solution by quadratures, it is possible to rewrite (52) in the form

$$Q = 1 + \frac{s_\alpha^{(1)}(x)}{\alpha} \left(\int_{x_0}^x e^{\frac{1}{\alpha} \int_y^x s_\alpha^{(1)}(z) dz} dy + C e^{\frac{1}{\alpha} \int_{x_0}^x s_\alpha^{(1)}(z) dz} \right).$$

and (59) in the form

$$Q^v = \int_{x_0}^x e^{\frac{1}{\alpha} \int_y^x s_\alpha^{(1)}(z) dz} dy + C e^{\frac{1}{\alpha} \int_{x_0}^x s_\alpha^{(1)}(z) dz}.$$

Here C is an arbitrary constant. Since a linear combination of symmetries is a symmetry, both terms on the RHS are by themselves the characteristics of symmetries. The symmetry associated with the factor multiplying C is the nonlocal symmetry first presented in [56]. The relation between Bäcklund transformations and nonlocal symmetries has recently been discussed in [42].

A conservation law (CL) for a PDE for the scalar function $u(x, t)$ is an expression

$$T_t + X_x = 0$$

which holds on solutions of the equation. Conservation laws for CH can be obtained from (13) by writing it in the form

$$s_t + (su - \alpha(u_x + 2s))_x = 0 \quad (61)$$

Using the expansion (53) for s in (61) we obtain an infinite hierarchy of conservation laws. Terms with integer powers of α in this expansion give trivial CLs. To prove this, observe from (54) that the terms with integer powers are obtained by setting $s = \frac{1}{2} (s_\alpha^{(1)} + s_\alpha^{(2)})$ in (61). But from (11) it is simple to verify that

$$\frac{1}{2} (s_\alpha^{(1)} + s_\alpha^{(2)}) = -\alpha (\ln |s_\alpha^{(1)} - s_\alpha^{(2)}|)_x .$$

Thus to obtain nontrivial laws we look at only the half-integer powers of α . Thus we set $s = s_\alpha^{(1)} - s_\alpha^{(2)}$ in (61) to obtain the generating conservation law

$$T = s_\alpha^{(1)} - s_\alpha^{(2)} , \tag{62}$$

$$X = (u - 2\alpha)(s_\alpha^{(1)} - s_\alpha^{(2)}) . \tag{63}$$

The expansion of the generating conservation law around $\alpha = 0$ gives an infinite hierarchy of nontrivial CLs for CH and pCH. The first few take the form

$$\begin{aligned} T_1 &= \sqrt{m} , \\ X_1 &= uT_1 , \\ T_2 &= \frac{1}{8m^{5/2}} (4m^2 + 4mm_{xx} - 5m_x^2) , \\ X_2 &= uT_2 - 2T_1 , \\ T_3 &= \frac{1}{128m^{11/2}} (64m^3m_{xxxx} - 1105m_x^4 + 1768mm_x^2m_{xx} - 304m^2m_{xx}^2 - 448m^2m_xm_{xxx} \\ &\quad - 96m^3m_{xx} + 200m^2m_x^2 - 16m^4) , \\ X_3 &= uT_3 - 2T_2 . \end{aligned}$$

Similar results can be obtained for HS and the full family of equations (19). The existence of an infinite number of conservation laws for CH follows from the bihamiltonian structure for CH discovered in [9]. The local form of these conservation laws was first obtained in [23], and they were subsequently further studied, and their derivation simplified, in numerous works such as [56, 57, 28, 25, 27, 35, 11, 30, 24]. The derivation given here can be easily related to previous ones, though the use of 2 solutions of (11)-(12) to understand the triviality of “half” of the conservation laws, is, we believe, new.

8 Concluding remarks

In this paper we have explored the theory of the Bäcklund transformation for the Camassa-Holm equation. This is an unfamiliar type of BT, as it acts on one of the independent variables, as well as the dependent variables. However, it has emerged that it is just as useful — using the superposition principles for the action on the different variables, we can exploit the BT to write down two wave solutions, just as is done for standard integrable equations such as KdV. Furthermore, we have shown how a double BT encodes an infinite set of symmetries for CH, and the relationship of the BT and conservation laws.

We have seen that the BT can also generate “unphysical” solutions, by which we mean solutions for which the new independent variable is not a 1 – 1 function of the old independent variable. Going beyond two wave solutions, it is not clear exactly what superpositions are allowed without creating singularities, though it seems to be a reasonable hypothesis that all possible combinations of solitons and cuspons can be formed,

with the speeds permitted by the value of u_0 , as listed in the table at the end of section 5. It seems to us that this is a problem that remains to be handled independent of the method used for constructing multiwave solutions.

Peakons emerge from both solitons and cuspons in the limit $u_0 \rightarrow 0$ (with one giving rise to peakons of positive speed and one to peakons of negative speed, depending on whether the limit is taken from below or from above). This is an extremely singular limit. We have not yet found a way to apply a superposition principle directly to peakons, but we continue to search.

Finally, one more general comment. The BT, in its minimalist form, is the transformations (10) and (14) where s satisfies (11)-(12). The latter equations for s are equivalent to the Lax pair, or linear system, for CH. So the BT seems to be more than the linear system. We wonder if there is a case of an integrable system without a BT?

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